1 Quantum Zeno

In this problem, you will study a version of the quantum Zeno effect – where repeated projective measurements can slow down the dynamics of a quantum system.

(a) The zeno effect arises from the fact that in quantum mechanics amplitudes, not probabilities, grow linearly in time. Consider a state \( |\Psi(0)\rangle = |g\rangle \) evolving under Hamiltonian \( \hat{H} = \Omega |e\rangle\langle g| + h.c. \) What is \( |\Psi(\delta t)\rangle \) for very small \( \delta t \)?

(b) Suppose at \( t = \delta t \) a measurement is made in the \( |e\rangle, |g\rangle \) basis. What is the probability that the system will be found in the ground state?

(c) Now suppose that \( |\Psi\rangle \) evolves under \( \hat{H} \) for a time \( t = N\delta t \), and that at each times \( t_k = k\delta t \) a measurement is made in the \( |e\rangle, |g\rangle \) basis, for \( k = 1, ..., N \). What is the probability \( P_N \) that the system will be found in the ground state at each timestep \( k \) (and in particular, at \( t = N\delta t \))?

(d) Let \( N \to \infty \) while keeping \( N\delta t = \text{constant} \). What is \( \lim_{N \to \infty} P_N \)? Could you reproduce this result in any kind of classical decay process?

2 Conservative potential (semiclassical)

The conservative potential for a two-level atom arising from a far-detuned laser beam can be estimated as

\[
U \approx \frac{\hbar \Omega^2}{4\delta},
\]

where \( \delta = \omega_L - \omega_{21} \) is the detuning between the laser frequency and the internal atomic transition.

(a) Justify this estimate using a dressed state picture for light-matter interaction.

(b) Assume a Gaussian beam with waist \( w_0 = 1\text{mm} \). Assume you have transition wavelength of 780nm and a laser with 1064nm. Further assume that \( \gamma = 2\pi \cdot 6\text{MHz} \). How much laser power do you need in order to trap a particle that has total energy \( \approx k_B \cdot 1\text{mK} \)?

(c) Estimate the off-resonant photon scattering rate for this set of parameters.

(d) Calculate the conservative potential (in RWA) arising from two counter-propagating laser beams that form a standing wave with electric field

\[
E(x, t) = E_0 \cos(xk - \omega_L t) + E_0 \cos(-xk - \omega_L t)
\]

Sketch the resulting potential energy.

(e) How is the potential modified if you consider a Gaussian beam profile? Assume that both beams are focused at the same position and approximate the potential around the focus position.
3 Doppler cooling

In the lecture we have discussed the dipole force induced by a laser beam. Additionally there are dissipative forces arising from scattering of photons. For a plane wave with wave vector \( k \), the atoms experiences a recoil kick with momentum \( \hbar k \), when it absorbs a photon. The resulting force is \( F = \hbar k \Gamma \), where \( \Gamma \) is the scattering rate. For atoms moving at a certain velocity \( \vec{v} \) we also have to include the Doppler shift \( k \vec{v} \). Taking this frequency shift into account the dissipative force can be expressed as

\[
\vec{F} = \frac{\hbar k \gamma}{2} \frac{s_0}{1 + s_0 + 4 \frac{(\delta - k \vec{v})^2}{\gamma^2}}.
\]  (3)

(a) Justify this expression using heuristic arguments.

(b) What is the absolute maximum of the force that you could reach with infinite laser power? Estimate this force in units of multiples of \( g \) (acceleration from gravity at earth surface) for a Sodium atom. Assume again \( \gamma \approx 2\pi \cdot 6\text{MHz} \)

(c) Calculate the Doppler force from two counter-propagating beams in the low saturation limit. Sketch your result for blue and red detuning (i.e., for positive and negative detuning) as a function of velocity.

(d) Show that the force can be linearized for small velocities:

\[
\vec{F} = -\beta \vec{v}.
\]  (4)

Which detuning maximizes the friction coefficient \( \beta \).