

Ultracold quantum systems, Problem sheet no 1

A reminder of your atomic physics course

Prof. Immanuel Bloch

Wintersemester 2009/2010

1 Angular Momentum

A three component hermitian operator $\vec{\mathbf{L}}$ with the components \mathbf{L}_i which satisfies the commutation relations $[\mathbf{L}_i, \mathbf{L}_j] = i\hbar \sum_k \epsilon_{ijk} \mathbf{L}_k$ is called an angular momentum operator.

1.1 angular momentum in QM

The magnitude of $\vec{\mathbf{L}}$ is given by $\mathbf{L}^2 = \sum_i \mathbf{L}_i^2$ and $\mathbf{L}_\pm := \mathbf{L}_x \pm i\mathbf{L}_y$. Show that

- $[\mathbf{L}^2, \mathbf{L}_i] = 0$
- $[\mathbf{L}_z, \mathbf{L}_\pm] = \pm\hbar\mathbf{L}_\pm$
- $\mathbf{L}^2 = \mathbf{L}_z^2 + \frac{1}{2}(\mathbf{L}_+\mathbf{L}_- + \mathbf{L}_-\mathbf{L}_+)$

1.2 addition of spin, clebsch-gordan coefficients

Consider a system with two angular momenta $\vec{\mathbf{L}}_1$ and $\vec{\mathbf{L}}_2$, Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. In many cases (where total angular momentum is conserved) it is convenient to use the total angular momentum $\vec{\mathbf{L}} := \vec{\mathbf{L}}_1 + \vec{\mathbf{L}}_2$ with eigenstates $|l_1 l_2 L M\rangle$ for operators \mathbf{L}_1^2 , \mathbf{L}_2^2 , \mathbf{L}^2 and \mathbf{L}_z

- Show that \mathbf{L} is an angular momentum operator
- Clebsch-Gordan coefficients The eigenstates of the combined angular momentum can be expressed in terms of the original eigenfunctions of the original basis like this

$$|j_1 j_2 J M\rangle = \sum_{j_1 j_2 m_1 m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle$$

the coefficients for this expansion are called Clebsch Gordan coefficients, which in turn can be expressed in terms of the Wigner 3j-symbols like so

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-)^{j_1 - j_2 + M} \sqrt{2J + 1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$$

some main properties are summarized here:

- the 3j symbols are real
- triangular inequality: $|l_1 - l_2| \leq L \leq |l_1 + l_2|$
- angular momentum projection conservation: $m_1 + m_2 + m_3 = 0$

- invariant under cyclic permutation of columns
- permutation of two columns multiply by $(-1)^{l_1+l_2+l_3}$ same factor if all m 's change sign.
- orthogonal: $\sum_{m_1, m_2} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L' \\ m_1 & m_2 & -M' \end{pmatrix} = \frac{\delta_{L, L'} \delta_{M, M'}}{3L+1}$

Calculate total spin eigenfunctions of two spin 1/2 particles.

Hint: the 3j-Symbols for the case of at least one spin 1/2 particle are

$$\begin{pmatrix} l & 1/2 & l-1/2 \\ m & \mu & -m-\mu \end{pmatrix} = -\frac{(-1)^{-l-m} \sqrt{\frac{(l-m)!(l+m)!}{l(2l+1)(\frac{1}{2}-\mu)!(l-m-\mu-\frac{1}{2})!(\mu+\frac{1}{2})!(l+m+\mu-\frac{1}{2})!}}}{\sqrt{2}} \quad (1)$$

$$\begin{pmatrix} l & 1/2 & l+1/2 \\ m & \mu & -m-\mu \end{pmatrix} = -\frac{i(-1)^{l+m+\mu} \sqrt{\frac{(l-m-\mu+\frac{1}{2})!(l+m+\mu+\frac{1}{2})!}{(2l^2+3l+1)(l-m)!(l+m)!(\frac{1}{2}-\mu)!(\mu+\frac{1}{2})!}}}{\sqrt{2}} \quad (2)$$

Literature: e.g. Messiah, Quantum mechanics

c) test

2 Atom in an external electric field. Perturbation theory

Assume an atom with hamiltonian \mathbf{H} and known energy eigenstates $|\psi_n\rangle$ so that $\mathbf{H}_0 |\psi_n\rangle = E_n |\psi_n\rangle$. Now the atom is placed into a homogeneous external electric field so that the system hamiltonian becomes

$$\mathbf{H} = \mathbf{H}_0 + e\vec{E} \cdot \vec{r}$$

calculate the lowest order correction to the eigen energy E_n in perturbation theory, assuming that this energy is non-degenerate. Hint: the dipolemoment of $|\psi_n\rangle$ vanishes. Do you know why?

3 atomic clocks

In a cesium atomic clock a transition between the F=3 and F=4 hyper fine level of the cesium atom is employed. The energy level of states in an external magnetic field are described by the Breit-Rabi-formula

$$E_{J=1/2, m_J, I, m_I} = -\frac{\Delta E_{hfs}}{2(2I+1)} + g_I \mu_B m_I B \pm \frac{\Delta E_{hfs}}{2} \left(1 + \frac{3m_I x}{2I+1} + x^2 \right)^{1/2}$$

with $\Delta E_{hfs} = A_{hfs}(I+1/2)$ ($A_{hfs} = h2.2981579425$ GHz exact) the hyper fine splitting, $m = m_I \pm 1/2$ (the sign being as in the formula) and

$$x = \frac{(g_J - g_I)\mu_B B}{\Delta E_{hfs}}$$

. For cesium g_I is small compared to g_J and can be neglected.

- sketch the position of the energy levels as a function of the magnetic field
- why is the transition $m_F = 0$ to $m_F = 0$ chosen as a clock transition?
- calculate the transition frequency of this transition as a function of magnetic field.
- what is the maximum allowable field if the transition frequency is to differ by less than 10^{-12} from the unperturbed frequency.