Ultracold quantum systems, Problem set no 8
Josephson effect

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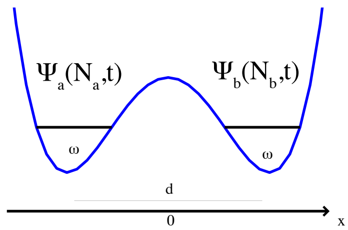
8.1 Double Well



Consider a one dimensional double well potential as sketched in the figure. The separation between the minima is d and the trap frequency in each well is ω.

In the limit of a noninteracting Bose gas, a large separation between the two potential minima and a high barrier between them, the ground state wave function for each individual minimum is the harmonic oscillator wavefunction $\Psi_{a,b}$ for the wells a and b with corresponding particle numbers N_a and N_b.

(a) Calculate the particle current (using the standard probability current formula) at the symmetry point of the potential ($x = 0$) for a wavefunction

$$\Psi = \Psi_a + \Psi_b e^{i\phi}. \tag{1}$$

as a function of N_a, N_b and ϕ.

(b) For which separation d between the two minima does the current $j/\sqrt{N_a N_b}$ become comparable to the trap frequency?

8.2 SQUIDs

Superconductors can be understood as Bose-Einstein condensates of the electron gas in the conductor. Of course electrons have spin 1/2 and are therefore fermions. Thus in order to Bose-Einstein condense, the electrons have to pair up. These pairs constitute the fundamental particles in the condensate with the corresponding Gross-Pitaevskii (GPE) equation[1] for these particles reads

$$i\hbar\partial_t\Psi = \left[\frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{2e}{c}\mathbf{A}\right)^2 + \alpha + \beta|\Psi|^2\right]\Psi \tag{2}$$

where $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Note that Eigen functions (including the ground state) for $\mathbf{A} = 0$ are real functions.

[1] in this context, this equation is called the Landau-Ginzburg equation

8.2.1 AC-Josephson effect

Two such superconductors are coupled using a tunnel junction. A constant voltage is applied between these two. Calculate the current as a function of time and voltage (assuming a characteristic tunnel current I_J).

8.2.2 gauge invariance

Show that if Ψ is a solution to the GPE equation, so is $\Psi'(\mathbf{r},t) = \Psi(\mathbf{r},t)e^{i\phi(\mathbf{r})}$ with adjusted $\mathbf{A}' = \mathbf{A} - \frac{\hbar c}{2e}\nabla\phi$. This gauge transformation does not change the magnetic field (do you know why?).

8.2.3 closed wire loop

Now lets look at a closed superconductiong wire loop of length L parameterized by the length along the loop l. If the equations of motion are restricted to this loop, they will read

$$i\hbar\partial_t\Psi(l,t) = \left[\frac{1}{2m}\left(-i\hbar\partial_l - \frac{2e}{c}A_l(l)\right)^2 + \alpha + \beta|\Psi(l,t)|^2\right]\Psi(l,t). \tag{3}$$

where now A_l is the projection of $\mathbf{A}$ onto the loop tangent (so that $\oint_{loop}\mathbf{A}d\dot{\mathbf{l}} = \int_0^L A_l(l)dl$

(a) What are the quantization levels for the magnetic flux through the loop (use Stokes' theorem)? What are the corresponding currents? Hint: use that wave functions must be continous and apply stokes theorem.

(b) can the flux through the loop be changed without breaking super conductivity?

(c) now we break the loop with a tunnel junction. how does the loop current depend on the magnetic flux through the loop (again assuming a Josephson current I_J)?