Ultracold quantum systems, Problem set no 7
Vortices and Solitons

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Assume we have a dilute gas in a trap. The Gross-Pitaevskii equation of the system is given by

$$-i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left(-\frac{\hbar^2}{2m}\triangle+V(\mathbf{r})+g|\Psi|^2(\mathbf{r},t)\right)\Psi(\mathbf{r},t)=:\hat{H}\Psi(\mathbf{r},t). \tag{1}$$

7.1 Rotating Gases

(a) Assume we start rotating the potential with an angular frequency Ω around the z-axis of the coordinate system. Show, that the hamiltonian $\hat{H}'$ in the rotating frame takes the form

$$\hat{H}'=\hat{H}-\Omega\hat{L}_z \tag{2}$$

(b) Assuming a noninteracting ($g=0$) gas in an (almost) isotropic harmonic potential with a trapping frequency ω, at which rotation frequency will the ground state of the system develop a vortex? Hint: look at the states with nonzero orbital angular momentum L_z. Is rotation above this critical rotation frequency stable?

(c) The $\hat{L}=1,\hat{L}_z=1$ state of the harmonic oscillator is given by (up to normalization)

$$\Psi(x)=(x+iy)e^{-\frac{x^2+y^2}{2a_0^2}} \tag{3}$$

where a_0 is the width of the ground state wave packet. Sketch the density distribution of this state. Now the trapping potential is suddenly switched off. Calculate how the wave packet expands in time. Does the vortex survive expansion? Does it change it's shape?

(d) The kinetic energy of a vortex can be estimated by

$$E_{kin}=\frac{m\bar{n}}{2}\int_\xi^R v^2(\mathbf{r})d^2r \tag{4}$$

with ξ being the healing length, $\bar{n}$ an effective density and R the effective radius of the cloud. Give an expression for the kinetic energy of a vortex of charge n

(e) Assume a rotation with total charge n in a 'large' system (what is the scale for 'large'?). Will the ground state system have a single vortex of charge n or multiple vortices? If the latter case, how many?

7.2 Solitons

Show that the grey soliton wave function $\Psi(x,t)\propto i\frac{v_0}{c}+\sqrt{1-\frac{v_0^2}{c^2}}\tanh(k(x-v_0t))$ with $(k\xi)^2=1-\left(\frac{v_0}{c}\right)^2$, c being the speed of sound, is a solution to the Gross-Piaevskii without external potential equation. ξ is the healing length and v_0 is the velocity of the soliton.