Ultracold quantum gases
Problem set no 6

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This weeks exercise is a bit more technical and is intended to give you the chance to further familiarize yourself with both the notation of second quantization and the powerful method of Bogoliubov transformations.

1 Bogoliubov transformation

As you have heard in the lecture, the excitations of a BEC can be described -similar to condensed matter physics- by quasiparticles, the so-called Bogoliubov excitations.
Here we consider for simplicity a homogeneous BEC. As there is no explicit dependence on position one can simplify the calculations by using a momentum space representation. In addition we assume again weak interactions and low temperature, such that almost all atoms are in the condensate ($k = 0$): $N - N_0 << N$. In this regime one can decompose -as was discussed in the lecture- the field operators into a classical mean value and small deviations: $\hat{\Psi} = \Psi + \delta\hat{\Psi}$. Considering only terms up to first order in $\delta\hat{\Psi}$, expressing the hamiltonian in momentum space in second quantization results in:

$$\hat{H} = \frac{gN^2}{2V} + \sum_k \frac{\hbar^2 k^2}{2m} a_k^+ a_k + \frac{gN}{V} \sum_{k \neq 0} a_k^+ a_k + \frac{gN}{2V} \sum_{k \neq 0} \left(a_k^+ a_{-k}^+ + a_k a_{-k} \right)$$

where a_k (a_k^+) denotes the annihilation (creation) operator for a particle with momentum $\hbar k$, N is the total atom number, V the volume of the gas and g denotes the strength of the contact interaction. The first term in this hamiltonian accounts for the interaction energy of the condensate ($N \approx N_0$), the kinetic energy of the atoms is given by the second term and the third term describes the interaction between excited atoms with momentum k and the condensate (of density N/V).
A serious complication arises due to the fourth term: This term describes the creation (and annihilation) of k, $-k$ pairs due to collisions of two condensate atoms.

The standard way to diagonalize this hamiltonian is the so-called Bogoliubov transformation, which exchanges the creation and annihilation operators of the bosonic atoms a_k and a_k^+ by new operators b_k and b_k^+, that describe the annihilation and creation of Bogoliubov quasiparticles:

$$\begin{aligned} a_k &= u_k b_k + v_k b_{-k}^+ \\ a_k^+ &= u_k b_k^+ + v_k b_{-k} \end{aligned}$$

with u_k,v_k being scalar coefficients.

- As the new operators shall describe bosonic quasiparticles, we require them to obey bosonic commutation relations:

  $$[b_k, b_{k'}] = [b_k^+, b_{k'}^+] = 0 \qquad [b_k, b_{k'}^+] = \delta_{k,k'}$$

  Show that these commutation relations require the coefficients to obey:

  $$u_k^2 - v_k^2 = 1$$

- Show that the inverse Bogoliubov transformation -which expresses the Bogoliubov operators in terms of the free particle operators- is given by:

  $$\begin{aligned} b_k &= u_k a_k - v_k a_{-k}^+ \\ b_k^+ &= u_k a_k^+ - v_k a_{-k} \end{aligned}$$

- Write the Hamiltonian in the new Bogoliubov basis.

- The main advantage of the Bogoliubov picture is that the coefficients u_k, v_k can be chosen such that all non-diagonal terms, i.e. terms of the form $b_k^+ b_{-k}^+$ or $b_k b_{-k}$ vanish. Show that this is the case if the coefficients satisfy the condition:

  $$\left(\frac{\hbar^2 k^2}{2m} + ng \right) u_k v_k + \frac{n}{2} g \left(u_k^2 + v_k^2 \right) = 0$$

  where $n = N/V$ denotes the condensate density.

- Solve the above equation for the coefficients u_k and v_k under the condition $u_k^2 - v_k^2 = 1$. You should get the following result:

  $$\begin{aligned} u_k^2 &= \frac{1}{2} + \frac{1}{2\epsilon_k} \left(\frac{\hbar^2 k^2}{2m} + ng \right) \\ v_k^2 &= -\frac{1}{2} + \frac{1}{2\epsilon_k} \left(\frac{\hbar^2 k^2}{2m} + ng \right) \\ \epsilon_k &= \left[\left(\frac{\hbar^2 k^2}{2m} \right)^2 + \frac{n\hbar^2 k^2 g}{m} \right]^{1/2} \end{aligned}$$

- Write down the resulting hamiltonian and try to give an intuitive picture on it's meaning and structure.