Ultracold quantum systems, Problem set no 14
Quantum thermodynamics

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14.1 Thermodynamics of a fermionic spin mixture in a harmonic trap

Consider a harmonically trapped Fermi gas described by the grand canonical ensemble with two equally populated spin states. Between the two spin components exists a Feshbach resonance so that the scattering length a can be tuned from attractive to repulsive and bosonic molecules can be formed from pairs of the atoms.

14.1.1 Fermionic phase

For the ideal fermi gas, the grand canonical potential can be written as $\Omega = k_b T \int_0^\infty \mathrm{d}\epsilon \rho(\epsilon) \ln[1 - n(\epsilon)]$ where $n(\epsilon) = (\exp(\beta(\epsilon - \mu) + 1)^{-1}$ is the Fermi weighting factor, $\beta_F = (k_B T)^{-1}$ and $\rho(\epsilon)$ is the density of states. In the equally populated spin mixture in a harmonic trap the latter is given by $\rho(\epsilon) = \epsilon^2/(\hbar\omega)^3$. ($\omega$ being the trap frequency. compare to the formula given in the lecture)

(a) Show that in the above case the grand canonical potential is give by $\Omega = -1/3(\hbar\omega)^{-3} \int_0^\infty \mathrm{d}\epsilon \epsilon^3 n(\epsilon)$ and express it for the degenerate case $k_B T/\mu \ll 1$ using the Sommerfeld expansion $\int_0^\infty \mathrm{d}\epsilon H(\epsilon) n(\epsilon) = \int_0^\mu \mathrm{d}\epsilon H(\epsilon) + \sum_{k \text{ odd}} \frac{2H^{(k)}(\mu)}{\beta^{k+1} k!} \int_0^\infty \mathrm{d}x \frac{x^k}{e^x+1}$. (you may use a table to calculate the latter integrals)

(b) Calculate the entropy S and total atom number N from Ω and use the latter (after approximating it consistent with $k_B T \ll \mu$) to express S in terms of N, T and ω, dropping terms $o(T^3)$.

N.B. the entropy calculated like this is a good approximation for $k_F|a| < 1/2$.

14.1.2 Cooling the fermionic phase

It is difficult to reach really low temperatures with an interacting fermi liquid via evaporative cooling since the crutial thermalizing collisions are strongly suppressed as the temperature goes lower. This is not true for bosons. So how about cooling the bosonic molecules and then *adiabatically* returning to the fermionic case?

By using the appropriate state density for a bose gas in a harmonic trap $\rho_{mol}(\epsilon)$ (see e.g. Dalvovo et. al, Rev. Mod. Phys., **71**, p463 (1999)) one can, similar to the previous calculation determine the grand canonical potential and from that the entropy of the BEC at finite temperature. The density of states is slightly more complicated since interactions can not be neglected in this case. In a semiclassical approximation the result of this calculation is

$$S_{mol} \approx k_B N_{mol} \left(\frac{T}{T_{C,BEC}} \right)^3 \left(\frac{2\pi^4}{45\zeta(3)} + 3\frac{\mu_{mol}}{k_B T} \right), \tag{1}$$

with $\zeta(3) = 1.202$ the Riemann zeta function, N_{mol} the number of molecules and μ_{mol} the molecular chemical potential. This expression is accurate within 10% for $k_B T/\mu_{mol} \geq 1/10/$

(a) write down an expression for T/T_F when a molecular BEC is with temperature $T/T_{C,BEC}$ is adiabatically transferred into a ideal fermi liquid.

(b) Plot the resulting relative temperature $t_f = T/T_F$ as a function of the final relative BEC temperature $t_B = T/T_{C,BEC}$ for typical achievable values $t_B = 1 \ldots 0.25$ and different chemical potentials $\mu_{mod} = 1, 1/2, 1/4 k_B T_{BEC}$

14.2 Chandrasekhar limit

On the last problem set, the Fermi pressure of a completely degenerate fermi gas was calculated and found to be $P = (\hbar/5m_e)(6\pi^2/\alpha)^{2/3} n^{5/3}$, with m_e the mass of the electron, n the number density and α the degeneracy factor of the gas (2 for electrons). Now consider a star consisting of matter with μ_e nucleons of mass m_P per electron. When such a star has burnt all it's fuel, all atoms have masses around Fe, so that μ_e becomes ≈ 2. It cools down until the Fermi pressure becomes dominant. In this state the star is called a white dwarf. Then a balance between Fermi pressure and gravitational force determines the density distribution. Let's approximate the star by a homogeneous sphere. Then the gravitational energy is given by $E_G = -3/5 GM/R$, where G is the gravitational constant, M is the mass of the star and R it's radius. In equilibrium the total energy $E = U + E_G$, U being the ground state energy of the degenerate fermi gas, is minimized.

(a) Show that minimization of E is equivalent to $3pV = -E_G$.

(b) Write down the Fermi pressure as a function of the star's mass density ρ.

(c) Determine the radius of the star as a function of it's mass, by solving the equation in (a). Determine the mass, where the fermi energy becomes comparable to the electron rest mass and compare it to the mass of our sun. What is the fermi temperature T_F in this case?

(d) calculate the Fermi pressure in the ultra-relativistic limit $E(k) = \hbar kc$. In this limit, the radius of the star is no longer determined by energy minimization since the ground state energy U has the same dependence on R as the gravitational energy E_G. The gravitational pressure can no longer be balanced by the fermi pressure and the star can shrink without limit. However, the equation in (a) can still be solved for the mass. This is the Chandrasekhar mass limit. Calculate it and express it in units of the solar mass.

(e) Calculate the radius of the star as a function of it's mass numerically using the full relativistic energy-momentum relation and plot it together with the nonrelativistic limit.

N.B. the real density distribution of the star is determined by the 'hydrostatic equilibrium' condition

$$\frac{\partial P}{\partial r} = g\rho(r) = \frac{GM(r)}{r^2}\rho(r), \tag{2}$$

where $M(r)$ is the total mass of the distribution inside the radius r. The adventurous can attempt to solve this equation and find the more realistic density distribution.