Ultracold quantum gases
Problem set no 13

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1 Fermions in a box

We consider N indistinguishable Fermions in a 3D box of volume V and dimensions $L \times L \times L$ with periodic boundary conditions. The single-particle Eigenstates are given by plane waves of momentum

$$\psi_{\vec{p}} = \frac{1}{\sqrt{V}} e^{i\vec{p}x/\hbar}$$

and their Eigenenergies are given by $E_{\text{kin}} = \frac{p^2}{2m}$.

- Show that the choice of periodic boundary conditions implies that the allowed momenta are given by

$$\vec{p} = \frac{2\pi\hbar}{L}\vec{n} \qquad \vec{n} \in \mathbb{Z}^3$$

- Due to the Pauli principle N fermions at zero temperature will occupy the N lowest energy Eigenstates of the system. Show that the highest occupied momentum, the so-called *Fermi-momentum* $\hbar\, k_F$ is given by

$$\hbar\, k_F = \hbar \left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

  and calculate the Fermi energy.

- Show that the density of states $\rho(E)$ in this system is given by

$$\rho(E) = \frac{1}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} V\sqrt{E}$$

- Calculate the average energy per particle at zero temperature using the already calculated density of states. How does this energy depend on the size of the box? This dependence is known as "Pauli pressure" (or Fermi pressure): $P = -\frac{\partial E}{\partial V}$

2 Chemical potential

The number of particles as a function of chemical potential μ and temperature T is given by the following integral over the density of states time the Fermi-Dirac distribution:

$$N = \int_0^\infty \rho(E) N_{\mathrm{FD}}(E, \mu, T)\, dE$$

Calculate (numerically) the chemical potential needed to describe a gas of $N = 10$ atoms ($m = \hbar = L = k_B = 1$) at various temperatures. Try to give an intuitive explanation of the observed behavior in terms of the thermodynamical definition of the chemical potential! What is the chemical potential in the limit of zero temperature and how does it compare to the bosonic case?