Ultracold quantum systems, Problem set no 11
More double wells and coherent states

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11.1 Detuneable double well

As on the last problem set, consider a double well with on site interaction U and tunnel coupling J with a total atom number n. Assume now additionally, that the two wells can be detuned with respect to each other by an energy Δ, i.e. $\epsilon_R = -\epsilon_L = \Delta/2$.

(a) write down the hamiltonian (in 'bose hubbard' approximation) for this system for two and four particles in the atom number basis.

(b) Diagonalize the system numerically for both cases

(c) Plot the eigenenergies of the system as a function of detuning Δ for $U = 4J$.

(d) For the 4 atom case assume we start out strongly detuned (how much is strong?) with all the particles sitting in the lower/upper of the wells (or more precisely in the lowest/highest energy eigenstate). Then we start ramping the detuning to the other side adiabatically (how fast can we go in both cases?). Plot the expectation value of the occupation of one well as a function of detuning for $U = 30J$.

11.2 Distinguishable particles

now assume a situation as above, only with two particles that can be distinguished (e.g. two different spin states).

(a) write down the 'two species' bose hubbard hamiltonian for this system.

(b) Diagonalize the system numerically.

(c) calculate the time evolution of the state for a detuning $\Delta = U$ and initial conditions $|LR\rangle$ (denoting first atom sitting left and second right) and plot the probability for each particle to sit left for large $U >> J$. Discuss the result. What is the difference to the bosonic case above?

11.3 Coherent states, for those who care...

What follows is Quantum Optics text book knowledge. If you want to read more about coherent states you can for example look in Sakurai "Modern Quantum Mechanics" or Schleich "Quantum Optics in Phase Space".

11.3.1 Eigenstate of the destruction operator

A coherent state $|\alpha\rangle$ can be defined as an eigenstate to the destruction operator $\hat{a}$ (of the harmonic oscillator and the usual properties, frequency ω) with complex(!) eigenvalue α.

(a) From this property and using the (general) ansatz $|\alpha\rangle = \sum_{n=0}^{\infty} w_n |n\rangle$ where $|n\rangle$ is the n-th eigenstate of the harmonic oscillator, derive the coefficients w_n. Remember to normalize the state.

(b) write down the 'photon' number probability distribution $p(n)$ of this state, calculate the average 'photon' number and the variance.

(c) Determine the time evolution of the coherent state. Does it remain a coherent state? If yes, what is it's eigenvalue as a function of time?

11.3.2 Displaced 'Vacuum'

For the harmonic oscillator, the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ can be decomposed into the position and momentum operators like this

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \tag{1}$$

and similar for $a^\dagger$. We define the displacement operator $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger + \alpha^* \hat{a})$

(a) using $\langle x|p\rangle = (2\pi\hbar)^{-1/2} e^{ipx/\hbar}$ show that $\langle x| \exp(-ia\hat{p})|\psi\rangle = \langle x - a|\psi\rangle$, that is the operator $\exp(-ia\hat{p})$ displaces the wavefunction (in x representation) by a. It is easy to see that the operator $\exp(-ia\hat{x})$ has the same effect in p representation. What do these operators do in the respective other representations?

(b) Use the Baker-Hausdorff formula ($e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2}$ if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$) to write down the wave function

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle \tag{2}$$

both in momentum and in position space.

(c) calculate the same wave function in the eigenbasis of the harmonic oscillator $|\alpha\rangle = \langle n|\alpha\rangle |n\rangle$.