Ideal Bose Gas in a Trap

Density Matrix in the grand-canonical Ensemble

$$\boxed{\hat{\rho} = \frac{1}{\Xi} e^{-\beta(\hat{H} - \mu \hat{N})}}$$

with

$$\beta = \frac{1}{k_B T}.$$

Furthermore we introduce the fugacity z

$$z = e^{\beta\mu}$$

Harmonic Trapping Potential

Trapping potential $$U(\vec{r}) = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

Energy eigenvalues $$\varepsilon_l = l_x \hbar\omega_x + l_y \hbar\omega_y + l_z \hbar\omega_z$$

Mean occupation number of each eigenstate is given by the Bose distribution:

$$\boxed{n_l = \frac{1}{e^{\beta(\varepsilon_l - \mu)} - 1} = \left(\frac{1}{z} e^{\beta l \hbar\omega} - 1 \right)^{-1}}$$

Occupation number cannot become negative, therefore: $0 < z < 1$

Important Limits: $z \longrightarrow 1$ Bose-Einstein condensation

$z \longrightarrow 0$ Boltzmann statstics

Saturation of Population in the Excited States

How many particles occupy excited states ?

$$N' = \sum_{l\neq 0} n_l = \sum_{l\neq 0}\left(z^{-1}e^{\beta l\hbar\omega}-1\right)^{-1} < \sum_{l\neq 0}\left(e^{\beta l\hbar\omega}-1\right)^{-1} = N'_{\mathrm{max}}$$

→ For a given Temperature ***T*** and trapping parameters ω, only a maximum number of atoms can occupy the excited states of our trapping potential !

All remaining atoms $N-N'_{max}$ have to occupy the ground state of our trap !

↓

BEC

Critical Temperature and Number of Condensate Atoms

Onset of Bose-Einstein condensation at critical temperature:

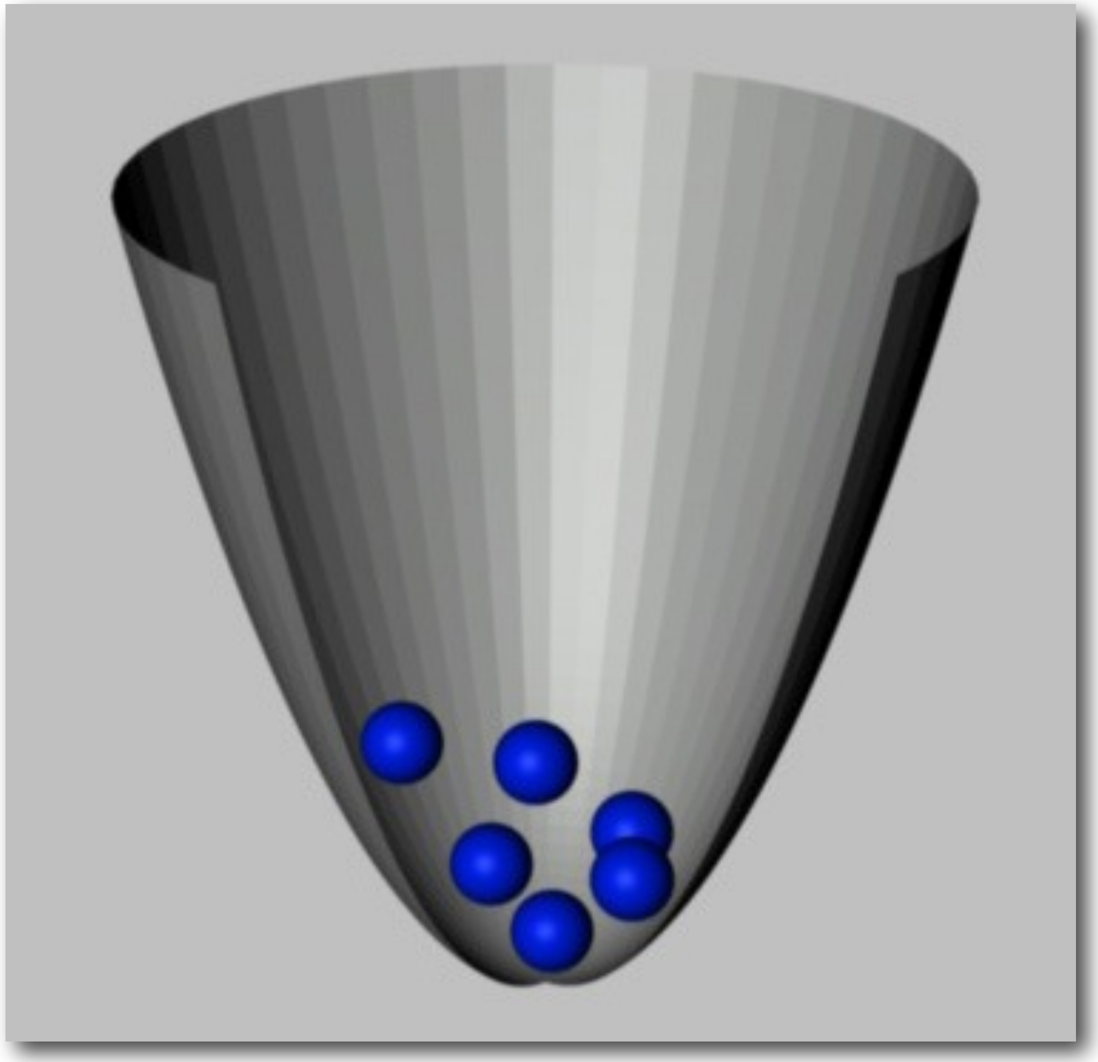
$$\boxed{k_B T_c = \hbar\omega \cdot \left(\frac{N}{\varsigma(3)} \right)^{1/3}}$$

Fraction of condensed atoms:

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^{3}}$$

cp. Homogeneous case: $\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2}$

How Can We Describe the Ground State of a Many-Body System in a Trap ?



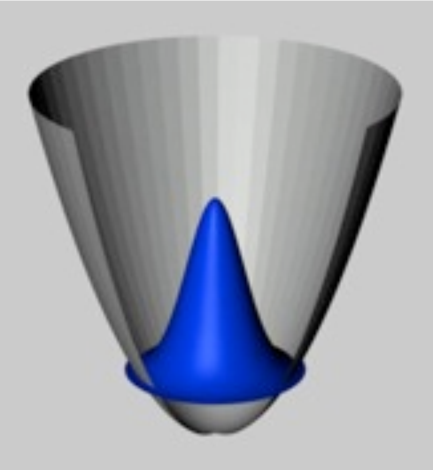
External confinement

N particle system

From a Bose Gas without Interactions to a Strongly Correlated Bose System

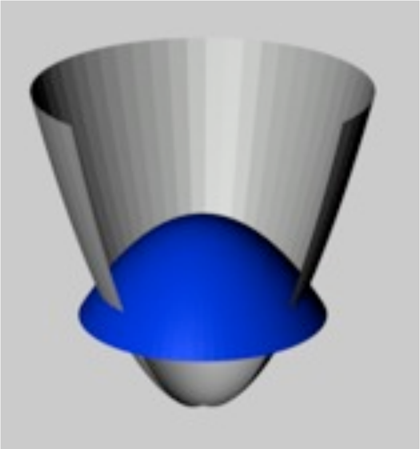
Many-Body State

No Interactions



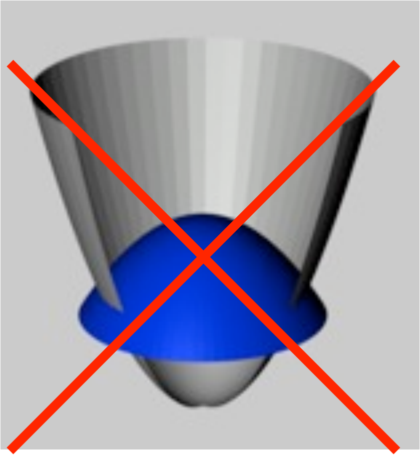
$$|\Psi\rangle \propto |\psi\rangle^{\otimes N}$$

Weak Interactions



$$|\Psi\rangle \propto |\psi_{\text{int}}\rangle^{\otimes N}$$

Strongly Correlated System



$$|\Psi\rangle \xcancel{\propto} |\psi_{\text{int}}\rangle^{\otimes N}$$

Macroscopic Wavefunction and Gross-Pitaevskii equation

Macroscopic wavefunction or order parameter:

$$\Psi(\vec{r}) = \sqrt{n(\vec{r})} \cdot e^{i\theta(\vec{r})}$$

This wavefunction can be obtained as a solution of a nonlinear Schrödinger equation, the Gross-Pitaevskii equation.

Chemical potential

$$-\frac{\hbar^2}{2m}\Delta\Psi(\vec{r}) + V(r)\Psi(\vec{r}) + \frac{4\pi\hbar^2 a}{m}\left|\Psi(\vec{r})\right|^2 \Psi(\vec{r}) = \mu\Psi(\vec{r})$$

Kinetic energy term

External potential term

Mean field
Term due to
interactions !

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External potential term

Mean field Term due to interactions !

Thomas-Fermi solution of the Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\Delta\Psi(\vec{r}) + V(r)\Psi(\vec{r}) + g\left|\Psi(\vec{r})\right|^2\Psi(\vec{r}) = \mu\Psi(\vec{r})$$

With large number of atoms with repulsive interactions (g>0), the macroscopic wavefunction is spread out due to the interactions, so that its curvature becomes very small.

Remember $g = \frac{4\pi\hbar^2 a}{m}$

Thomas-Fermi solution

$$\left|\Psi(\vec{r})\right|^2 = \frac{1}{g}\left(\mu - V(\vec{r})\right)$$

is good when $N\frac{a}{a_{ho}} \gg 1$

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Neglect Kinetic energy term !

Thomas-Fermi solution

$$\left|\Psi(\vec{r})\right|^2 = \frac{1}{g}\left(\mu - V(\vec{r})\right)$$

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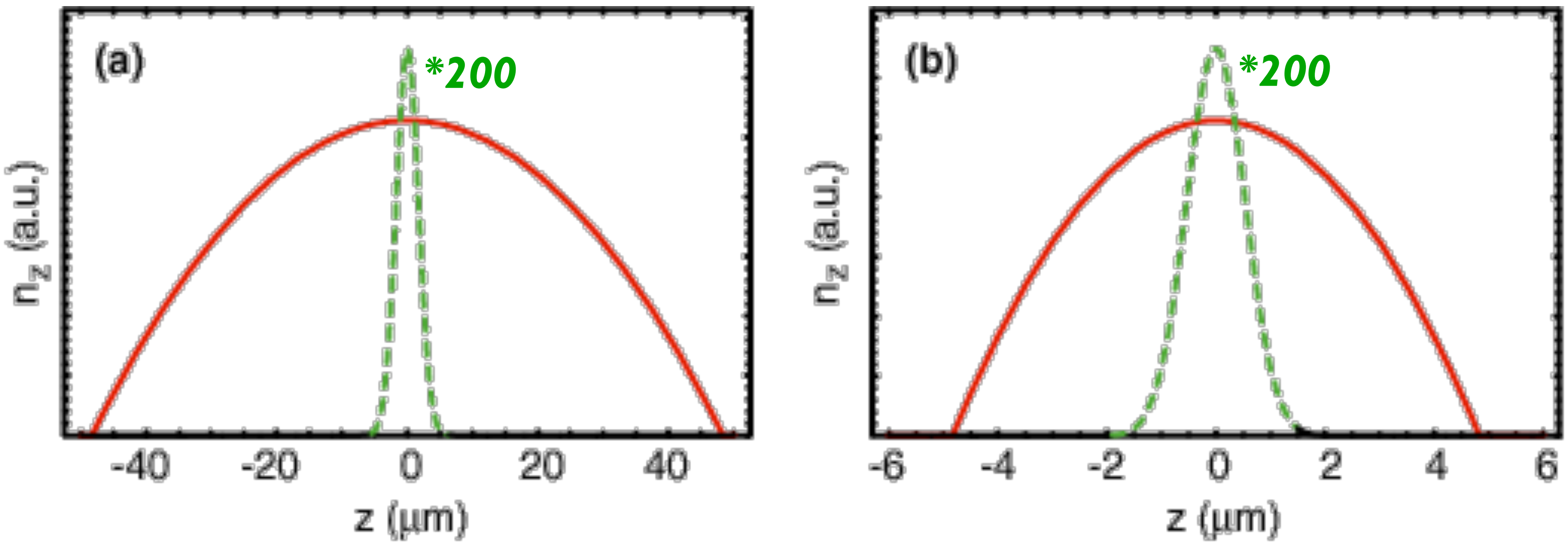
Thomas-Fermi solution compared to the Harmonic Oscillator Ground State

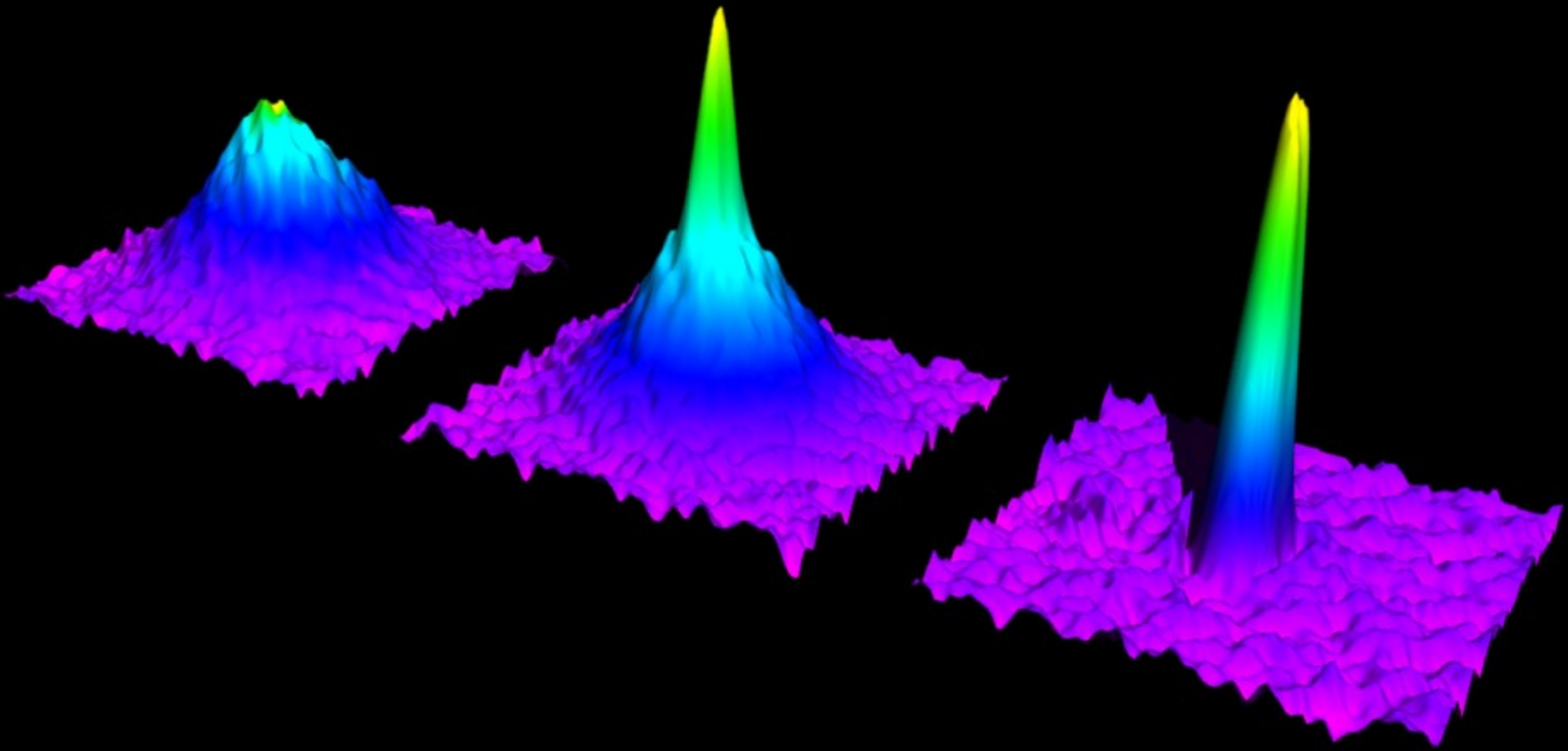


Axial profile ***Radial profile***

Parameters: ***10^6 ^{87}Rb atoms***

$$\omega_z = 2\pi \times 20\,\text{Hz}$$

$$\omega_r = 2\pi \times 200\,\text{Hz}$$



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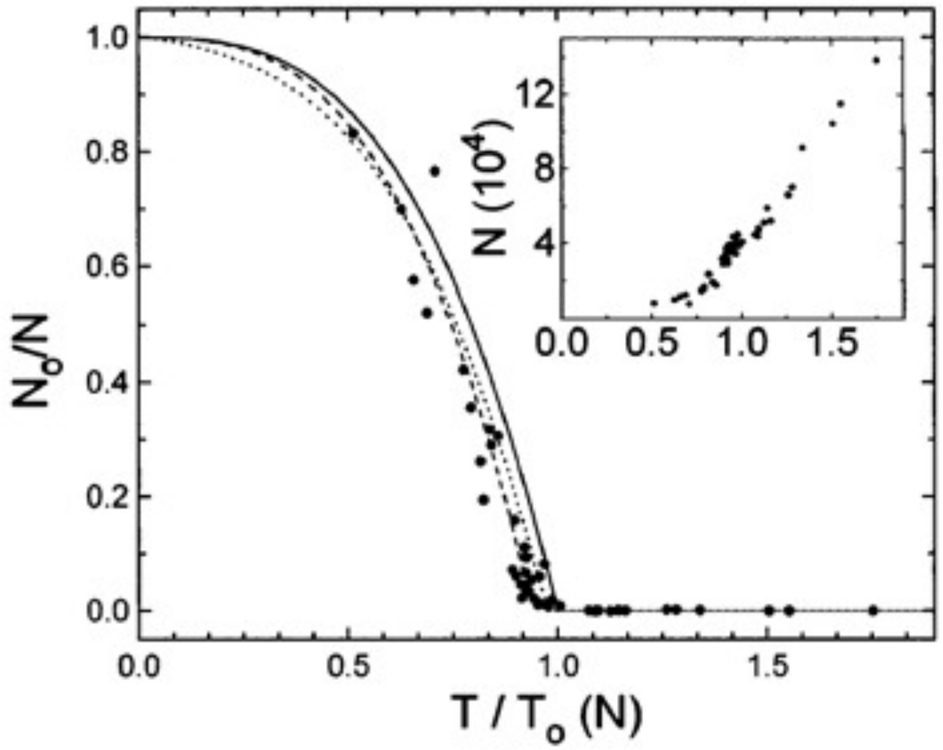
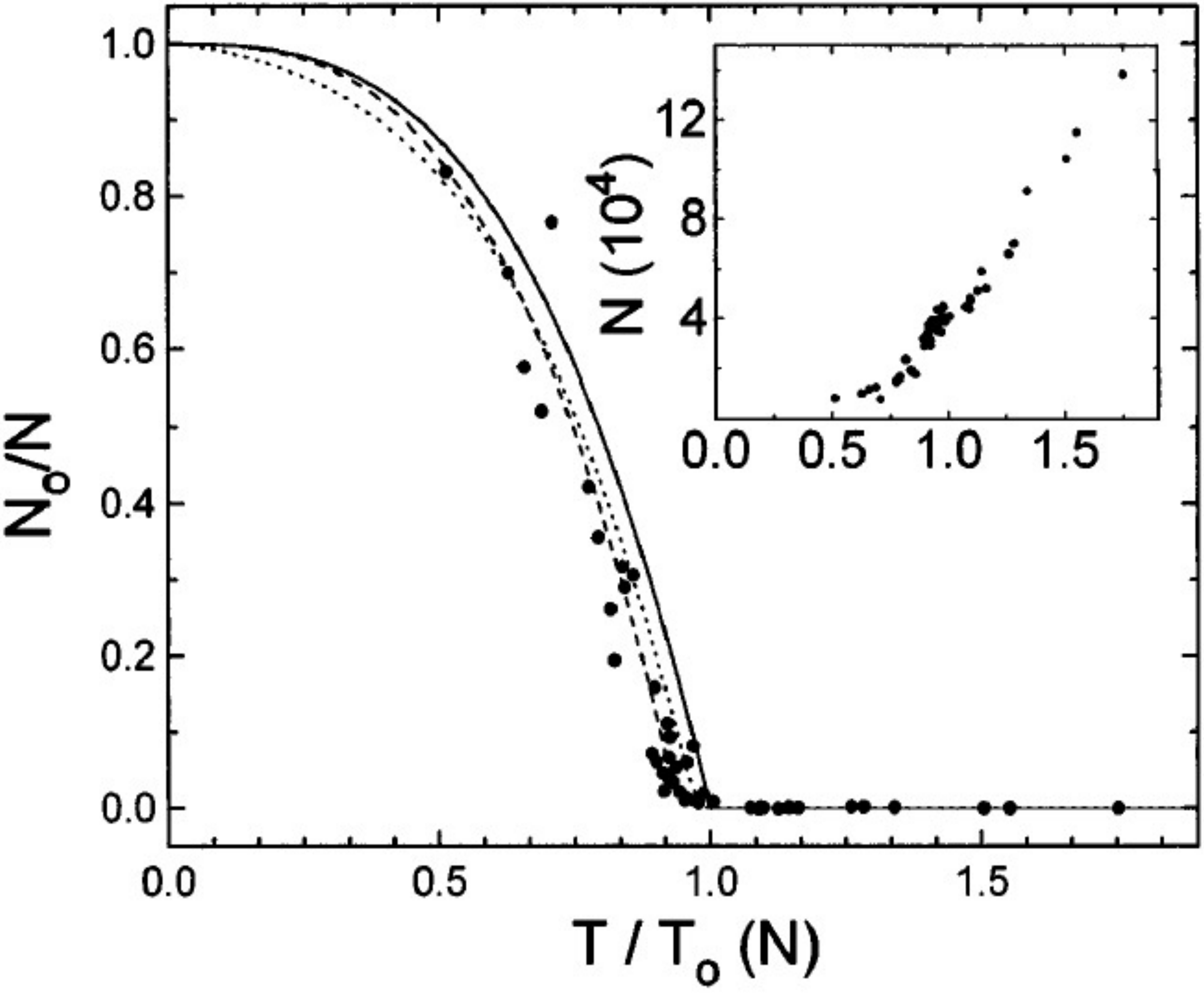
J. R. Ensher et al., PRL, (1996)





N_o/N
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
T / T_o (N)
N (10^4)
12
8
4
0.0
0.5
1.0
1.5